LENGTH SPECTRUM OF PERIODIC RAYS FOR BILLIARD FLOW

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Abstract. We study for several compact strictly convex disjoint obstacles the length spectrum $\mathcal{L}$ formed by the lengths of all primitive periodic reflecting rays. We prove the existence of sequences $\{\ell_j\}$, $\{m_j\}$ with $\ell_j \in \mathcal{L}$, $m_j \in \mathbb{N}$ such that the condition (LB) related to the dynamical zeta function $\eta_D(s)$ is satisfied. We construct such sequences under some separation condition for a small subset of $\mathcal{L}$ corresponding to lengths of the periodic rays with even reflexions. This separation condition is weaker than the assumption of an exponentially separated length spectrum $\mathcal{L}$. Moreover, we show that the periodic orbits in the phase space are exponentially separated.

Keywords: billiard flow, periodic reflecting rays, length spectrum, separation condition

1. Introduction

Let $D_1, \dots, D_r \subset \mathbb{R}^d$, $r \geqslant 3$, $d \geqslant 2$, be compact strictly convex disjoint obstacles with C^∞ smooth boundary and let $D = \bigcup_{j=1}^r D_j$. We assume that every D_j has non-empty interior and throughout this paper we suppose the following non-eclipse condition

$$D_k \cap \text{convex hull}\,(D_i \cup D_j) = \emptyset, \tag{1.1}$$

for any $1 \leqslant i, j, k \leqslant r$ such that $i \neq k$ and $j \neq k$. Under this condition all periodic trajectories for the billiard flow φ_t in $\Omega = \mathbb{R}^d \setminus \mathring{D}$ are ordinary reflecting ones without tangential intersections to the boundary ∂D. We consider the (non-grazing) billiard flow φ_t (see [CP, Section 2.2], [Pet25b, Section 2] for the definition) and the periodic trajectories will be called periodic rays. For any periodic ray γ, denote by $\tau(\gamma) > 0$ its period, by $\tau^\sharp(\gamma) > 0$ its primitive period, and by $m(\gamma)$ the number of reflections of γ at the obstacles. Denote by $\mathcal{P}$ the set of all oriented periodic rays and by P_γ, $\gamma \in \mathcal{P}$, the associated linearised Poincaré map (see [PS17, Section 2.3] for the definition). Consider the

Dirichlet dynamical zeta function

$$\eta_{\mathrm{D}}(s)=\sum_{\gamma\in\mathcal{P}}(-1)^{m(\gamma)}\frac{\tau^{\sharp}(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id}-P_{\gamma})|^{1/2}},\ \mathrm{Re}\, s\gg 1. \tag{1.2}$$

We have the estimates (see for instance [Pet99, Appendix])

$$C_1 e^{\mu_1\tau(\gamma)}\leq|\det(\mathrm{Id}-P_{\gamma})|\leq e^{\mu_2\tau(\gamma)},\ \forall\gamma\in\mathcal{P} \tag{1.3}$$

with constants $C_1>0,\ 0<\mu_1<\mu_2$. The series $\eta_D(s)$ is absolutely convergent and not vanishing for sufficiently large $\mathrm{Re}\, s$.

The zeta function $\eta_D(s)$ is important for the analysis of the distribution of the scattering resonances related to the Laplacian in $\mathbb{R}^d\setminus D$ with Dirichlet boundary conditions on ∂D. For more details we refer to [Ika90b, Section 1], [CP, Section 1]. It was proved in [CP, Theorem 1 and Theorem 4] that η_D admits a *meromorphic continuation* to $\mathbb{C}$ with simple poles and integer residues. There is a conjecture that η_D cannot be prolonged as *entire function*. This conjecture was established for obstacles with real analytic boundary in [CP, Theorem 3] and for obstacles with sufficiently small diameters [Ika90b], [Sto09] and C^∞ smooth boundary.

The difficulties to examine the analytic singularities of $\eta_D(s)$ are related to the change of signs of the coefficients of the Dirichlet series (1.2) which may produce cancellations. To study these cancelations, introduce the distribution

$$\mathcal{F}_{\mathrm{D}}(t)=\sum_{\gamma\in\mathcal{P}}\frac{(-1)^{m(\gamma)}\tau^{\sharp}(\gamma)\delta(t-\tau(\gamma))}{|\det(\mathrm{Id}-P_{\gamma})|^{1/2}}\in\mathcal{S}'(\mathbb{R}^+).$$

Let $\psi\in C_0^\infty(\mathbb{R};\mathbb{R}_+)$ be an even function with $\mathrm{supp}\,\psi\subset[-1,1]$ such that $\psi(t)=1$ for $|t|\leqslant 1/2$. Let $(\ell_j)_{j\in\mathbb{N}}$ and $(m_j)_{j\in\mathbb{N}}$ be sequences of positive numbers such that $\ell_j\to\infty$, $m_j\to\infty$ as $j\to\infty$ and

$$\ell_j\geqslant d_0=2\min_{k\neq m}\mathrm{dist}\,(D_k,D_m)>0,\ m_j\geqslant\max\big\{1,\frac{1}{d_0}\big\}.$$

Define

$$\psi_j(t)=\psi(m_j(t-\ell_j)),\ t\in\mathbb{R}.$$

Definition 1.1. We say that the condition (LB) for F_D is satisfied if there exist constants $\alpha_0>0,\alpha_1>0,c_1>0$ such that for all $\beta\geq\alpha_1$ there exist sequences $(\ell_j),(m_j)$ with $\ell_j\nearrow\infty$ as $j\to\infty$ and $e^{\beta\ell_j}\leqslant m_j\leqslant\mathrm{e}^{2\beta\ell_j}$ satisfying

$$|\langle\mathcal{F}_{\mathrm{D}},\psi_j\rangle|\geqslant c_1\mathrm{e}^{-\alpha_0\ell_j},\ \forall j. \tag{1.4}$$

The estimate (1.4) gives exponentially small lower bounds for the sum of the contributions to $\langle F_D, \psi_j \rangle$ of the rays $\gamma \in \mathcal{P}$ with lengths

$$\tau(\gamma) \in (\ell_j - e^{-m_j}, \ell_j + e^{-m_j}),\ j \in \mathbb{N}.$$

If (LB) is satisfied, one obtains two important corollaries:

(i) The modified Lax-Phillips conjecture (MLPC) for scattering resonances introduced by Ikawa [Ika90a, page 212] holds. (MLPC) says that there exists a strip $\{z \in \mathbb{C} : 0 < \operatorname{Im} z \leq \alpha\}$ containing an infinite number of scattering resonances for Dirichlet Laplacian in $\mathbb{R}^d \setminus D$. For definition of scattering resonances and more precise results the reader may consult Chapter 5 in [LP89] for d odd and Chapter 4 in [DZ19]).

(ii) The function $\eta_D(s)$ has infinite number of poles in some strip $\{s \in \mathbb{C} : \operatorname{Re} s \geq \delta\}$ and we have a lower bound of the counting function of the poles in this strip (see [Pet25b, Theorem 1.1]). In fact, the result in [Pet25b, Theorem 1.1] has been stated assuming that $\eta_D(s)$ cannot be prolonged as an entire function, however the proof works if sequences (ℓ_j), (m_j) satisfying (1.4) exist.

On the other hand, Ikawa [Ika90a, Proposition 2.3] showed that if $\eta_D(s)$ cannot be prolonged as entire function, then (LB) holds for F_D. For obstacles with C^∞ boundary some conditions which imply that $\eta_D(s)$ cannot be prolonged as entire function have been established in [Pet25a]. It is interesting to find conditions leading to (LB) which are not related to the existence of poles of $\eta_D(s)$. In this paper we study this problem.

To construct sequences $\{\ell_j\}, \{m_j\}$ satisfying (1.4), we must study the distribution of the periods of periodic rays which has independent interest. Let $\Pi \subset \mathcal{P}$ be the set of primitive periodic orbits of billiard flow φ_t and let $\Pi_+ \subset \Pi$ (resp. $\Pi_- \subset \Pi$) be the set of periodic rays with even (resp. odd) number of reflexions. The counting function of the lengths satisfies

$$\sharp\{\gamma \in \Pi :\ \tau(\gamma) \leqslant x\} \sim \frac{\mathrm{e}^{hx}}{hx}, \quad x \to +\infty, \tag{1.5}$$

with some $h > 0$ (see for instance, [PP90, Theorem 6.9] for weak-mixing suspension symbolic flows and [Ika90a], [Mor91] for symbolic models related to billiard flow). Moreover, we have the asymptotics (see [Gio10, Theorem 2])

$$\sharp\{\gamma \in \Pi_\pm :\ \tau(\gamma) \leqslant x\} \sim \frac{\mathrm{e}^{hx}}{2hx}, \quad x \to +\infty. \tag{1.6}$$

Introduce the *length spectrum* $\mathcal{L} = \{\tau(\gamma) : \gamma \in \Pi\}$. We say that $\mathcal{L}$ is *exponentially separated* if there exists $\nu > 0$ such that for all $\ell,\ \ell' \in \mathcal{L}$

we have

$$|\ell - \ell'| \geq e^{-\nu \max\{\ell, \ell'\}} \text{ if } \ell \neq \ell'. \tag{1.7}$$

From Theorem 1.1 below it follows that if $\mathcal{L}$ is exponentially separated, then the condition (LB) holds.

We recall some positive and negative results concerning the exponential separation of length spectrum $\mathcal{L}$. For compact Riemannian manifolds M with negative curvatures the metrics for which $\mathcal{L}$ is not exponentially separated are topologically generic and dense for C^k, $k > 3$, topology (see [DJ16, Theorem 4.1]). On the other hand, Schenck proved in [Sch20, Theorem 1]) that the set of metrics for which $\mathcal{L}$ is exponentially separated is dense in $C^k, k \geq 2$, topology and (1.7) holds with $\nu = \nu_k > 0$ depending on k and the dynamical characteristic. However, $\nu_k \to +\infty$ as $k \to \infty$, so an approximation with C^∞ metrics having exponentially separated length spectrum is an open problem.

For billiard flow φ_t the lengths $\ell \in \mathcal{L}$ are rationally independent for generic obstacles (see [PS17, Theorem 6.2.3]). This result implies that generically there are gaps between the lengths of different periodic rays. However the estimates of these gaps and the existence of generic obstacles with exponentially separated $\mathcal{L}$ seems to be difficult open problem. In contrast to the metric case mentioned above, for obstacles we may perturb generically only the boundary and the rays in $\mathbb{R}^d \setminus \bar{D}$ are always union of linear segments. Consequently, a perturbation of the boundary is much more restrictive than the perturbations of a metric studied in [DJ16] and [Sch20]. In section 4 we prove that the periodic orbits in the phase space are exponentially separated. This is an analog of Proposition 2 in [Sch20]. This result could be considered as a first step in the analysis of the existence of exponentially small gaps in $\mathcal{L}$ for generic obstacles.

It is important to remark that in (1.4) are involved the contributions of the iterated rays with periods in the set $\{m\ell : \ell \in \mathcal{L}, m \geq 2\}$. Hence even in the case when $\mathcal{L}$ is exponentially separated, for the analysis of (LB) the terms in (1.4) related to these rays must be estimated. In this paper we show that a separation condition concerning a very small subsets of rays $\gamma \in \Pi_+$ implies (LB). Our main result is the following

Theorem 1.1. *Assume that there exist $\delta > 0, 0 < \rho < \min\{1, h^{-1}\}, c_0 > 5 - \frac{h\rho}{3}$ and a sequence $q_j \nearrow +\infty$ such that*

$$\sharp\Big\{\gamma \in \Pi_+ :\ q_j - \rho < \tau(\gamma) \leq q_j,$$

$$|\tau(\gamma) - \tau(\gamma')| \geq e^{-\delta \max\{\tau(\gamma), \tau(\gamma')\}},\ \forall \gamma' \in \Pi \setminus \{\gamma\}\Big\} \geq \frac{c_0 \rho e^{\frac{hq_j}{3}}}{8 q_j}. \tag{1.8}$$

Then the condition (LB) *is satisfied for* F_D.

In Lemma 3.1 we prove that for every small $\epsilon > 0$ and $q_j \geq C(\epsilon)$ we have the lower bound

$$\sharp\{\gamma \in \Pi_+ : \ q_j - \rho < \tau(\gamma) \leq q_j\} \geq (1-\epsilon)\frac{\rho e^{hq_j}}{8q_j},$$

while the separation assumption in (1.8) concerns only $\mathcal{O}\Big(\frac{\rho e^{\frac{hq_j}{3}}}{8q_j}\Big)$ rays. For this reason we say that a very small subsets of $\{\gamma \in \Pi_+, \ q_j - \rho < \tau(\gamma) \leq q_j\}$ must be exponentially separated. Moreover, in Theorem 1.1 there is not separation condition for the lengths of $\gamma \in \Pi_-$.

The paper is organised as follows. In Section 2 we obtain upper and lower bounds of the number of iterated rays with odd and even number of reflexions. In Section 3 one examines the number of lengths of periodic rays in small intervals $]q_j - \rho, q_j]$ and we prove Theorem 1.1. The exponential separation of periodic rays in phase space is studied in Section 4. Finally, in Section 5 we formulate some open problems for generic obstacles.

2. Estimation of the number of iterated rays

Clearly, $d_0 \leq \tau(\gamma)$, $\forall \gamma \in \mathcal{P}$. Given $q \gg 1$, introduce the counting functions of the periods of iterated rays

$$N_{odd}(q) = \sharp\{\gamma \in \Pi_- : \ (2k+1)\tau(\gamma) \leq q, \ k \in \mathbb{N}, \ k \geq 1\},$$

$$N_{even}(q) = \sharp\{\gamma \in \Pi : \ 2k\tau(\gamma) \leq q, \ k \in \mathbb{N}, \ k \geq 1\}.$$

Therefore for $q \geq 4d_0$

$$(2k+1)d_0 \leq (2k+1)\tau(\gamma) \leq q \tag{2.1}$$

implies $k \leq [\frac{q}{2d_0} - 1/2] = p_q, \ p_q \geq 1$. Thus in the definition of $N_{odd}(q)$ one has $1 \leq k \leq p_q$, while in $N_{even}(q)$ we have $1 \leq k \leq [\frac{q}{2d_0}]$. If $\gamma \in \Pi_-$, the number of reflexions $m(\gamma)$ of γ is odd and the iterated ray

$$\gamma_{2k+1} = \underbrace{\gamma \cup \gamma \cup ... \cup \gamma}_{(2k+1) \, times}$$

with length $(2k+1)\tau(\gamma)$ will have odd reflexions, too. Hence the contribution of γ_{2k+1} in (1.2) contains a negative factor $(-1)^{(2k+1)m(\gamma)}$.

Proposition 2.1. *Let* $0 < \epsilon < 1/4$ *be fixed. Then there exists* $B_\epsilon \gg 1$ *such that for* $q \geq B_\epsilon$ *we have*

$$(1-\epsilon)\frac{3e^{\frac{hq}{3}}}{2hd} < N_{odd}(q) \leq (1+\epsilon)\frac{3e^{\frac{hq}{3}}}{2hq}, \tag{2.2}$$

$$(1-\epsilon)\frac{2e^{\frac{hq}{2}}}{hd} < N_{even}(q) \leq (1+\epsilon)\frac{2e^{\frac{hq}{2}}}{hq}. \tag{2.3}$$

Proof. Write

$$N_{odd}(q) = \sum_{k=1}^{p_q} \sharp\Big\{\gamma \in \Pi_- : \tau(\gamma) \leq \frac{q}{2k+1}\Big\}.$$

Applying (1.6), there exists $C_\epsilon > d_0 + 1$ such that for $x \geq C_\epsilon$ we have

$$(1-\frac{\epsilon}{2})\frac{e^{hx}}{2hx} \leq \sharp\{\gamma \in \Pi_\pm : \tau(\gamma) \leq x\} \leq (1+\frac{\epsilon}{2})\frac{e^{hx}}{2hx}. \tag{2.4}$$

We fix C_ϵ and choose $q \geq B_\epsilon > \max\{5C_\epsilon, 4d_0\}$. We have the sum

$$N_{odd}(q) = \sum_{[\frac{q}{C_\epsilon}] \geq 2k+1 \geq 3} \sharp\Big\{\gamma \in \Pi_- : \tau(\gamma) \leq \frac{q}{2k+1}\Big\}$$

$$+ \sum_{[\frac{q}{C_\epsilon}] < 2k+1 \leq \frac{q}{d_0}} \sharp\Big\{\gamma \in \Pi_- : \tau(\gamma) \leq \frac{q}{2k+1}\Big\} = J_1(q) + J_2(q).$$

There exists a constant $A_\epsilon > 1$ such that

$$\sharp\Big\{\gamma \in \Pi_- : \tau(\gamma) \leq C_\epsilon\Big\} \leq A_\epsilon.$$

According to (2.1) and (2.4), one deduces

$$J_1(q) \leq (1+\frac{\epsilon}{2})\frac{3e^{\frac{hq}{3}}}{h}\Big(\frac{1}{2q} + \frac{1}{3}\sum_{k=2}^{m(\epsilon,q)} \frac{e^{\frac{hq}{2k+1} - \frac{hq}{3}}}{d_0}\Big)$$

$$\leq (1+\frac{\epsilon}{2})\frac{3e^{\frac{hq}{3}}}{h}\Big(\frac{1}{2q} + \frac{m(\epsilon,q)-1}{3d_0}e^{-\frac{2hq}{15}}\Big),$$

where

$$m(\epsilon,q) = \begin{cases} [\frac{1}{2}[\frac{q}{C_\epsilon}] - 1/2] \text{ if } \frac{1}{2}[\frac{q}{C_\epsilon}] - 1/2 \notin \mathbb{N}, \\ \frac{1}{2}[\frac{q}{C_\epsilon}] - 1/2 \text{ if } \frac{1}{2}[\frac{q}{C_\epsilon}] - 1/2 \in \mathbb{N}. \end{cases}$$

Notice that $q \geq 5C_\epsilon$ implies $m(\epsilon,q) \geq 2$. Since in $J_2(q)$ one has $2k+1 \geq [\frac{q}{C_\epsilon}] + 1 > \frac{q}{C_\epsilon}$, we obtain

$$J_2(q) \leq (p_q - m(\epsilon,q))A_\epsilon.$$

Increasing B_ϵ, if it is necessary, one arranges for $q \geq B_\epsilon$ the inequalities

$$\frac{1}{2q} + \frac{m(\epsilon,q)-1}{3d_0}e^{-\frac{2hq}{15}} \leq \frac{1}{2q} + \frac{\epsilon}{8q(1+\epsilon/2)} = \frac{4+3\epsilon}{8q(1+\epsilon/2)},$$

$$(p_q - m(\epsilon,q))A_\epsilon \leq \frac{3\epsilon e^{\frac{hq}{3}}}{8hq}.$$

Combining the above estimates for $J_k(q),\ k = 1, 2$, we conclude that

$$N_{odd}(q) \leq \frac{(1+\epsilon)3e^{\frac{hq}{3}}}{2hq}.$$

To obtain the left hand side part of (2.2), we apply (2.4) and taking into account only the term

$$\sharp\{\gamma \in \Pi_- : \tau(\gamma) \leq q/3\},$$

one has

$$(1-\epsilon)\frac{3e^{\frac{hq}{3}}}{2hq} < (1-\frac{\epsilon}{2})\frac{3e^{\frac{hq}{3}}}{2hq} \leq N_{odd}(q).$$

For the proof of (2.3) we apply a similar argument and we omit the details. □

3. Length spectrum in small intervals

To estimate the number of periodic rays in Π_+ with lengths in a interval $(q-\rho, q]$, we need the following

Lemma 3.1. *Let $0 < \rho < \min\{1, h^{-1}\}$ and let $0 < \frac{2\epsilon}{1-\epsilon} \leq \frac{\rho h}{4}$. Then for $q \geq C(\epsilon)$ we have*

$$(1-\epsilon)\frac{\rho e^{hq}}{8q} \leq \sharp\{\gamma \in \Pi_+ : q-\rho < \tau(\gamma) \leq q\} \leq (5-h\rho)(1+\epsilon)\frac{\rho e^{hq}}{8q}. \quad (3.1)$$

Proof. An application of (1.6) with $q \geq C(\epsilon)$ yields

$$\sharp\{\gamma \in \Pi_+ :\ q-\rho < \tau(\gamma) \leq q\} \geq (1-\epsilon)\frac{e^{hq}}{2hq} - (1+\epsilon)\frac{e^{h(q-\rho)}}{2h(q-\rho)}$$

$$= (1-\epsilon)\frac{e^{hq}}{2hqe^{h\rho}}\Big(e^{h\rho} - \frac{(1+\epsilon)q}{(1-\epsilon)(q-\rho)}\Big).$$

Next, choosing $C(\epsilon)$ large enough, we obtain

$$\frac{(1+\epsilon)q}{(1-\epsilon)(q-\rho)} = \Big(1 + \frac{2\epsilon}{1-\epsilon}\Big)\Big(1 + \frac{\rho}{q-\rho}\Big)$$

$$\leq 1 + \frac{\rho h}{4} + \frac{\rho}{q-\rho}\Big(1 + \frac{2\epsilon}{1-\epsilon}\Big) \leq 1 + \frac{\rho h}{4} + \frac{\rho^2 h^2}{32} < e^{\frac{h\rho}{4}}.$$

This implies

$$e^{h\rho} - \frac{(1+\epsilon)q}{(1-\epsilon)(q-\rho)} > e^{h\rho}(1 - e^{-\frac{3h\rho}{4}}).$$

On the other hand, we have the inequality $f(y) = 1 - e^{-3y} - y \geq 0$ for $0 \leq y \leq \frac{\log 3}{3}$ because

$$f'(y) \geq 0 \text{ for } 0 \leq y \leq \frac{\log 3}{3}.$$

Therefore $\rho < \frac{1}{h} < \frac{4 \log 3}{3h}$ yields $\frac{h\rho}{4} < \frac{\log 3}{3}$, hence

$$1 - e^{-\frac{3h\rho}{4}} \geq \frac{h\rho}{4},$$

and we obtain the left hand side of (3.1).

To establish the upper bound in (3.1), notice that for $q \geq C(\epsilon)$ one has

$$\sharp\{\gamma \in \Pi_+ : \, q - \rho \leq \tau(\gamma) \leq q\} \leq (1+\epsilon)\frac{e^{hq}}{2hq} - (1-\epsilon)\frac{e^{h(q-\rho)}}{2h(q-\rho)}$$

$$= (1+\epsilon)\frac{e^{hq}}{2hq}\Big(1 - \Big(1 - \frac{2\epsilon}{1+\epsilon}\Big)\Big(1 + \frac{\rho}{q-\rho}\Big)e^{-h\rho}\Big).$$

Since $e^{-x} \geq 1 - x$ for $x \geq 0$, and $\frac{2\epsilon}{1+\epsilon} < \frac{h\rho}{4}$, we obtain

$$1 - \Big(1 - \frac{2\epsilon}{1+\epsilon}\Big)\Big(1 + \frac{\rho}{q-\rho}\Big)e^{-h\rho} \leq 1 - (1 - \frac{h\rho}{4})(1 - h\rho)$$

$$= h\rho\Big(\frac{5 - h\rho}{4}\Big).$$

This completes the proof. □

It is important to note that in the estimates (3.1) one has as factor the length of the interval $[q-\rho, q]$. Introduce

$$N_{odd}(q-\rho, q) = N_{odd}(q) - N_{odd}(q-\rho).$$

Clearly, $h\rho < 1$ implies $h\rho/3 < 1$. Exploiting (2.2), we obtain the following

Lemma 3.2. *Under the assumptions of Lemma* 3.1 *for* $q \geq C_\epsilon$ *we have*

$$(1-\epsilon)\frac{\rho e^{\frac{hq}{3}}}{8q} \leq N_{odd}(q-\rho, q) \leq (5 - \frac{h\rho}{3})(1+\epsilon)\frac{\rho e^{\frac{hq}{3}}}{8q}. \tag{3.2}$$

We apply (2.2) and get

$$N_{odd}(q) - N_{odd}(q-\rho) \leq (1+\epsilon)\frac{3e^{\frac{hq}{3}}}{2hq} - (1-\epsilon)\frac{3e^{\frac{h}{3}(q-\rho)}}{2h(q-\rho)}.$$

Next the proof is a repetition of that of Lemma 3.1 and we omit the details.

Proof of Theorem 1.1. First we choose $0 < \epsilon < 1$ small enough to arrange $c_0 > \frac{1}{3}(15 - h\rho + \epsilon)(1+\epsilon)$, $\frac{2\epsilon}{1-\epsilon} < \frac{rh}{4}$. Fix ϵ and consider the interval

$$(q_j - \rho - e^{-\delta q_j}, q_j + e^{-\delta q_j}] = (p_j - \rho_j, p_j]$$

with $p_j = q_j + e^{-\delta q_j}$ and $\rho_j = \rho + 2e^{-\delta q_j}$. Taking q_j large enough, one gets $\rho_j < \min\{1, h^{-1}\}$. We apply the upper bound in (3.2) for $N_{odd}(p_j - \rho_j, p_j)$ with $q_j \geq C(\epsilon)$ and deduce

$$N_{odd}(p_j - \rho_j, p_j) \leq \frac{15 - h\rho_j}{3}(1+\epsilon)\frac{\rho_j e^{\frac{hp_j}{3}}}{8p_j}. \tag{3.3}$$

We claim that for $q_j \geq m(\epsilon) \geq C(\epsilon)$ large we have

$$(15 - h\rho_j)\frac{\rho_j e^{\frac{h}{3}e^{-\delta q_j}}}{p_j} < (15 - h\rho + \epsilon)\frac{\rho}{q_j}. \tag{3.4}$$

This inequality is equivalent to

$$\Big(1 - \frac{e^{-\delta q_j}}{p_j}\Big)\Big(1 + \frac{2e^{-\delta q_j}}{\rho}\Big)e^{\frac{h}{3}e^{-\delta q_j}} < 1 + \frac{2he^{-\delta q_j} + \epsilon}{15 - h\rho_j}.$$

For $q_j \to +\infty$ the left hand side of the above inequality goes to 1, so for large q_j it will be less than $1 + \frac{\epsilon}{15 - h\rho} < 1 + \frac{\epsilon}{15 - h\rho_j}$. This proves the claim. Consequently, for $q_j \geq m(\epsilon)$ the estimate (3.4) implies

$$N_{odd}(p_j - \rho_j, p_j) \leq \frac{1}{3}(15 - h\rho + \epsilon)(1+\epsilon)\frac{\rho e^{\frac{hq_j}{3}}}{8q_j}.$$

Increasing $m(\epsilon)$ and taking into account (1.8), for $q_j \geq m(\epsilon)$ we obtain

$$\sharp\Big\{\gamma \in \Pi_+ : q_j - \rho < \tau(\gamma) \leq q_j, |\tau(\gamma) - \tau(\gamma')| \geq e^{-\delta \max\{\tau(\gamma), \tau(\gamma')\}}, \forall \gamma' \in \Pi \backslash \{\gamma\}\Big\}$$

$$\geq \frac{c_0 \rho e^{\frac{hq_j}{3}}}{8q_j} > \frac{1}{3}(15 - h\rho + \epsilon)(1+\epsilon)\frac{\rho e^{\frac{hq_j}{3}}}{8q_j} \geq N_{odd}(p_j - \rho_j, p_j).$$

This means that the number of rays $\gamma \in \Pi_+$ with $q_j - \rho < \tau(\gamma) \leq q_j$ such that the intervals

$$J_{\delta,j}(\gamma) = (\tau(\gamma) - e^{-\delta q_j}, \tau(\gamma) + e^{-\delta q_j})$$

contain only one $\tau(\gamma)$ with $\gamma \in \Pi$ is greater than $N_{odd}(p_j - \rho_j, p_j)$. Hence there exists $\gamma_j \in \Pi_+$ with $q_j - \rho < \tau(\gamma_j) \leq q_j$ such that $J_{\delta,j}(\gamma_j)$ does not contain the lengths of periodic rays $\gamma' \in \mathcal{P} \setminus \Pi$ having odd number of reflexions. On the other hand, some lengths of iterated rays with even number of reflexions could be in the interval $J_{\delta,j}(\gamma_j)$.

We choose $\ell_j = \tau(\gamma_j),\ \beta = \delta,\ m_j = e^{\delta \ell_j}$. Then in the interval $L_j = (\ell_j - m_j^{-1}, \ell_j + m_j^{-1})$ we have only lengths of periodic rays with even number of reflections and $\psi_j(\ell_j) = 1$. By using (1.3), we conclude that

$$\langle F_D, \psi_j \rangle = \sum_{\tau(\gamma) \in L_j} \tau^\sharp(\gamma) |\det(\mathrm{Id} - P_\gamma)|^{-1/2} \psi_j(\tau(\gamma)) \geq d_0 e^{-\frac{\mu_2}{2} \ell_j}.$$

This completes the proof of Theorem 1.1.

4. Separation of periodic orbits in phase space

We start with some preparations. Set

$$d_1 = \max_{k \neq j} \mathrm{dist}\,(D_k, D_j),\ d_2 = \frac{2 d_1}{d_0} \geq 1.$$

We recall some notations concerning billiard flow φ_t (see for more details [CP, Section 2]). Let $S\mathbb{R}^d$ be the unit tangent bundle of $\mathbb{R}^d$ and let $\pi : S\mathbb{R}^d \to \mathbb{R}^d$ be the natural projection. For $x \in \partial D_j$, denote by $n_j(x)$ the inward unit normal vector to ∂D_j at x pointing into D_j. Set $D = \bigcup_{j=1}^r D_j$ and

$$\mathcal{D} = \{(x, v) \in S\mathbb{R}^d\ :\ x \in \partial D\}.$$

Define the grazing set $\mathcal{D}_{\mathrm{g}} = T(\partial D) \cap \mathcal{D}$ and denote by $\langle .,. \rangle$ the scalar product in $\mathbb{R}^d$. We say that $(x, v) \in T_{\partial D_j} \mathbb{R}^d$ is incoming (resp. outgoing) if $\langle v, n_j(x) \rangle > 0$ (resp. $\langle v, n_j(x) \rangle < 0$). Introduce

$$\mathcal{D}_{\mathrm{in}} = \{(x, v) \in \mathcal{D}\ :\ (x, v) \text{ is incoming}\},$$
$$\mathcal{D}_{\mathrm{out}} = \{(x, v) \in \mathcal{D}\ :\ (x, v) \text{ is outgoing}\}.$$

For $(x, v) \in \mathcal{D}_{\mathrm{in/out/g}}$ denote by $v' \in \mathcal{D}_{\mathrm{out/in/g}}$ the image of v by the reflexion R_x with respect to $T_x(\partial D)$ at $x \in \partial D$, that is

$$v' = v - 2\langle v, n_j(x) \rangle n_j(x), \quad v \in S_x \mathbb{R}^d, \quad x \in \partial D_j.$$

The billiard flow $(\phi_t)_{t \in \mathbb{R}}$ is a complete flow acting on $S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$ which is defined as follows. For $(x, v) \in S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$ we set

$$\tau_\pm(x, v) = \pm \inf\{t \geqslant 0 : x \pm tv \in \partial D.\}$$

By convention, we have $\tau_\pm(x, v) = \pm\infty$, if the ray $x \pm tv$ has no common point with ∂D for $\pm t > 0$. For $(x, v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\mathrm{g}}$ we define

$$\phi_t(x, v) = (x + tv, v), \quad t \in [\tau_-(x, v), \tau_+(x, v)],$$

while for $(x, v) \in \mathcal{D}_{\mathrm{in/out}}$, we set

$$\phi_t(x, v) = (x + tv, v) \quad \text{ if } \quad \begin{cases} (x, v) \in \mathcal{D}_{\mathrm{out}},\ t \in [0, \tau_+(x, v)]\,, \\ \text{or } (x, v) \in \mathcal{D}_{\mathrm{in}},\ t \in [\tau_-(x, v), 0]\,, \end{cases}$$

and

$$\phi_t(x,v) = (x+tv', v') \quad \text{ if } \quad \begin{cases} \quad (x,v) \in \mathcal{D}_{\rm in},\ t \in \,]0, \tau_+(x,v)]\,, \\ \text{or } (x,v) \in \mathcal{D}_{\rm out},\ t \in [\tau_-(x,v'), 0[\,. \end{cases}$$

Introduce the non-grazing billiard table M as

$$M = B/\sim, \quad B = S\mathbb{R}^d \setminus \Big(\pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\rm g}\Big)\,,$$

where $(x,v) \sim (y,w)$ if and only if $(x,v) = (y,w)$ or

$$x = y \in \partial D \quad \text{ and } \quad w = v'.$$

The set M is endowed with the quotient topology.

The non-grazing flow φ_t is defined on M as follows. For $(x,v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\rm in}$ we set

$$\varphi_t([(x,v)]) = [\phi_t(x,v)], \quad t \in \,]\tau_-^{\rm g}(x,v), \tau_+^{\rm g}(x,v)[\,,$$

where $[z]$ denotes the equivalence class of $z \in B$ for the relation $\sim$, and

$$\tau_\pm^{\rm g}(x,v) = \pm \inf\{t > 0 : \phi_{\pm t}(x,v) \in \mathcal{D}_{\rm g}\}.$$

Notice that $\tau_\pm^{\rm g}(x,v) \neq 0$ for $(x,v) \in \mathcal{D}_{\rm in}$, while it is possible to have $\tau_\pm^{\rm g}(x,v) = \pm\infty$. The above formula defines a flow on M since each $(x,v) \in B$ has a unique representative in $(S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})) \cup \mathcal{D}_{\rm in}$. Therefore φ_t is continuous, but the flow trajectory of the point (x,v) for times $t \notin \,]\tau_-^{\rm g}(x,v), \tau_+^{\rm g}(x,v)[$ is not defined. The flow φ_t is defined for all $t \in \mathbb{R}$ for z in the trapping set K formed by points $z \in M$ such that $-\tau_-^{\rm g}(z) = \tau_+^{\rm g}(z) = +\infty$ and

$$\sup A(z) = -\inf A(z) = +\infty, \text{ when } A(z) = \{t \in \mathbb{R} : \pi(\varphi_t(z)) \in \partial D\}.$$

(see for more details see [CP, Section 2]).

Set $\overline{\mathcal{D}}_{\rm in} = \{(x,v) :\ x \in \partial D, |v| = 1, \langle v, n(x)\rangle \geqslant 0\}$ and define the *billiard ball map*

$$\mathbf{B} : \overline{\mathcal{D}}_{\rm in} \ni (x,v) \longmapsto (y,w) \in \overline{\mathcal{D}}_{\rm in},$$

where

$$(y,w) = (x + \tau_+^{\rm g}(x, R_x v) R_x v, R_x v),$$

and $R_x : v \in S_x\mathbb{R}^d \to v' \in S_x\mathbb{R}^d$ is the reflection with respect to $T_x(\partial D)$. The map $\mathbf{B}(x,v)$ is defined if $\tau_+^{\rm g}(x, R_x v) < +\infty$.

Consider a point $\rho = (x,v) \in \mathring{B}$. Assume γ is reflecting ray with $p \geq 1$ reflexions starting at ρ and going to

$$\phi_t(\rho) = \Big(\phi_\sigma \circ \mathbf{B}^p \circ \phi_\tau\Big)(\rho) \in \mathring{B},\ \tau > 0,\ \sigma > 0.$$

The map $\phi_\tau : \mathring{B} \to \mathcal{D}_{\rm in}$ is smooth near ρ and the map $\mathbf{B} : \overline{\mathcal{D}}_{in} \to \overline{\mathcal{D}}_{in}$ is also C^∞ smooth [Kov88]). We have the diagram

$$\begin{array}{ccc} \mathring{B} & \xrightarrow{\phi_t} & \mathring{B} \\ \Big\downarrow \phi_\tau & & \Big\uparrow \phi_\sigma \\ \mathcal{D}_{\rm in} & \xrightarrow{R \circ \mathbf{B}^p} & \mathcal{D}_{\rm out} \end{array}$$

and $\mathrm{d}\phi_t = \mathrm{d}\phi_\sigma \circ \mathrm{d}R \circ \mathrm{d}\mathbf{B}^p \circ \mathrm{d}\phi_\tau$. Here R denotes the reflection map $R : \, (y, w) \in \mathcal{D}_{\rm in} \to (y, R_y w) \in \mathcal{D}_{\rm out}$.

We can estimate

$$\|\mathrm{d}\mathbf{B}\|_{T(\partial D) \to T(\partial D)} \leq A_0$$

with constant $A_0 > 1$ depending of d_1 and the sectional curvatures of ∂D (see [CP, Appendix A]). Setting $\beta = 2 \log A_0 / d_0$, one deduces

$$\|\mathrm{d}\mathbf{B}^p\| \leq A_0^p = e^{\beta p d_0/2} < e^{\beta t},$$

where $t > p d_0/2$ is the length of γ. Thus we obtain the estimate

$$\|\mathrm{d}\phi_t(\rho)\|_{T(\mathring{B}) \to T(\mathring{B})} \leq C_0 e^{\beta t} \tag{4.1}$$

with $C_0 \geq 1$, $\beta > 0$ independent of ρ, τ, σ and p. Here $\|.\|$ is the norm induced on $T(\mathring{B})$ by the standard norm in $S\mathbb{R}^d$. We increase the constant β, if it is necessary, so that

$$e^{-\beta d_0/2} \eta_0 < d_0/2, \; e^{-\beta d_0} < 1/2. \tag{4.2}$$

In the case

$$\phi_t(\rho) = \Big(\mathbf{B}^p \circ \phi_\tau\Big)(\rho) \in \bar{\mathcal{D}}_{\rm in}$$

we obtain also the estimate

$$\|\mathrm{d}\phi_t(\rho)\|_{T(\mathring{B}) \to T(\partial D)} \leq C_0 e^{\beta t}, \tag{4.3}$$

exploiting the smoothness of the map $\mathbf{B} : \bar{\mathcal{D}}_{\rm in} \to \bar{\mathcal{D}}_{\rm in}$. Finally, the estimates (4.1), (4.3) hold in the more general case when $\phi_\tau(\rho) = (y, \eta) \in \mathcal{D}_g$. To prove this, consider a strictly convex surface Ω passing trough x with interior normal $-v$ at x. Then $-v$ is incoming direction and $(x, -v) \in T(\Omega)_{\rm in}$, where we define $T(\Omega)_{\rm in}$ in analogy with $\mathcal{D}_{\rm in}$. Then $\phi_\tau(\rho)$ coincides with the local billard ball map $\tilde{\mathbf{B}} : \, T(\Omega)_{\rm in} \to \bar{\mathcal{D}}_{\rm in}$ and this makes possible to apply the smoothness of $\tilde{\mathbf{B}}$.

Notice that every periodic reflecting ray γ is determined by a configuration

$$\alpha_\gamma = (i_1, \ldots, i_k),$$

where $i_j \in \{1, \ldots, r\}$, with $i_k \neq i_1$, $i_j \neq i_{j+1}$ for $j = 1, ..., k-1$ and α_γ is such that γ has *successive reflections* on $\partial D_{i_1}, \ldots, \partial D_{i_k}$. The configuration α_γ is well defined modulo cyclic permutation. We say that γ has type α_γ and α_γ has length k. Moreover, according to [PS17, Corollary 2.2.4], for a fixed configuration α_γ there exists at most one periodic ray γ in $\mathbb{R}^d \setminus \mathring{D}$.

Given a periodic ray γ in $\mathbb{R}^d \setminus \mathring{D}$, define by $\tilde{\gamma}$ one of the two possible lifts

$$\tilde{\gamma} = \{\varphi_t(x, \pm v) \in M:\ 0 \le t < \tau(\gamma),\ x \in \gamma,\ x \notin \partial D\}$$

on M, where $v \in \mathbf{S}^{d-1}$ is the direction of γ at x. Below we fix a lift $\tilde{\gamma} = \tilde{\gamma}(t)$ corresponding to (x, v) and parametrised by the length. Set

$$\mathcal{G}(T) = \{\tilde{\gamma}:\ \pi(\tilde{\gamma}) = \gamma \in \Pi,\ \tau(\gamma) \le T\}.$$

A point $z \in \mathring{B}$ will be called *linearly connected* to $\tilde{\gamma}$ if there exists $w \in \tilde{\gamma} \cap \mathring{B}$ such that $\sigma z + (1-\sigma)w \in \mathring{B}$, $\forall \sigma \in [0,1]$. For such points $z \in \mathring{B}$ define

$$\text{dist}(z, \tilde{\gamma}) = \min\{\|z - w\|:\ w \in \tilde{\gamma} \cap \mathring{B}, \sigma z + (1-\sigma)w \in \mathring{B},\ \forall \sigma \in [0,1]\},$$

$$\Theta^{\epsilon}_{\tilde{\gamma}} = \{z \in \mathring{B}:\ z \text{ is linearly connected to } \tilde{\gamma},\ \text{dist}(z, \tilde{\gamma}) \le \epsilon\}.$$

We will prove the following result.

Theorem 4.1. *There exists $\epsilon_0 > 0$ depending of C_0, d_0 and η_0 such that for any different periodic rays $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathcal{G}(T)$ we have*

$$\Theta^{\epsilon_0 e^{-(1+d_2)\beta T}}_{\tilde{\gamma}_1} \cap \Theta^{\epsilon_0 e^{-(1+d_2)\beta T}}_{\tilde{\gamma}_2} = \emptyset.$$

Proof. Choose $\epsilon_0 = \min\{\frac{\eta_0}{2C_0}, \frac{d_0}{2C_0}\}$. Let $\tilde{\gamma}_k = \tilde{\gamma}_k(t) \in \mathcal{G}(T)$, $k = 1, 2$, be two different periodic rays with configurations α_k having lengths p_k, respectively. Let $\tilde{\gamma}_k$ have periods $T_k \le T$, $k = 1, 2$ and let $\alpha_1 = (i_1, ..., i_{p_1})$. Assume that

$$\Theta^{\epsilon_0 e^{-(1+d_2)\beta T}}_{\tilde{\gamma}_1} \cap \Theta^{\epsilon_0 e^{-(1+d_2)\beta T}}_{\tilde{\gamma}_2} \neq \emptyset. \tag{4.4}$$

Then there exist points $\rho_k = (x_k, \xi_k) \in \mathring{B} \cap \tilde{\gamma}_k$, $k = 1, 2$, and $\rho = (\mu, \xi) \in \mathring{B}$ such that $\|\rho - \rho_k\| \le \epsilon_0 e^{-(1+d_2)\beta T}$, $k = 1, 2$ and

$$\nu_k(\sigma) = (1-\sigma)\rho_k + \sigma\rho \in \mathring{B},\ \sigma \in [0,1],\ k = 1, 2.$$

Assume that x_1 lies on a segment connecting $u_{i_{p_1}} \in \partial D_{i_{p_1}}$ and $u_{i_1} \in \partial D_{i_1}$, while x_2 lies on a segment connecting $w_{j_{q_1}} \in \partial D_{j_{q_1}}$ and $w_{j_1} \in \partial D_{j_1}$. If $i_1 = j_1$, since $\alpha_1 \neq \alpha_2$, there exist $m, q \in \{1, ..., r\}$, $m \neq q$, such that the ray $\tilde{\gamma}_1(t)$ issued from ρ_1 follows a configuration $\beta_1 = (i_1, ..., i_n, m)$, while the ray $\tilde{\gamma}_2(t)$ issued form ρ_2 follows the configuration $\beta_2 = (i_1, ..., i_n, q)$. The configurations β_k, $k = 1, 2$ may have the form $\beta_1 = (i_1, q, ...)$, $\beta_2 = (i_1, m, ...)$. Clearly, q and m are the first indices where we have difference. If $i_1 \neq j_1$ we have configurations

$\beta_1 = (i_1, ...)$, $\beta_2 = (j_1, ...)$. Below we treat the case $i_1 = j_1$. The analysis of the case $i_1 \neq j_1$ can be covered by a similar argument and we omit the details.

Without loss of generality we may assume that β_1, β_2 have length less or equal to p_1, that is $n \leq p_1 - 1$. Indeed, if

$$\beta_1 = (\underbrace{\alpha_1, ..., \alpha_1}_{\text{k times}}, i_1, ..., i_n, m),\ \beta_2 = (\underbrace{\alpha_1, ..., \alpha_1}_{\text{k times}}, i_1, ..., i_n, q),\ n \leq p_1 - 1,$$

we may cancel $\underbrace{\alpha_1, ..., \alpha_1}_{\text{k times}}$. For σ small enough the rays $\tilde{\gamma}_\sigma(t)$ issued from $\nu_1(\sigma)$ will follow the configuration β_1 with reflections on $\partial D_{i_1}, ..., \partial D_m$. In general the rays $\tilde{\gamma}_\sigma$ are not periodic, so after successive reflexions on $\partial D_{i_1}, ..., \partial D_m$ they may have other reflexions or glancing points and also they may escape to infinity. Let

$$\omega = \max\{\sigma \in [0, 1] : \tilde{\gamma}_\sigma(t) \text{ does not follow } \beta_1$$
$$\text{with reflections on } \ \partial D_{i_1}, ..., \partial D_{i_s}, \partial D_m\} > 0.$$

For the ray $\tilde{\gamma}_\omega(t)$ issued form $v_1(\omega)$ there two cases.

(a1). $\tilde{\gamma}_\omega(t)$ follows a configuration $\zeta = (i_1, ..., i_s)$ with reflections on $\partial D_{i_1}, ..., \partial D_{i_{s-1}}$ and tangency on ∂D_{i_s}, while

$$\beta_1 = (i_1, ..., i_s, ...),\ s \leq p_1 - 1.$$

(b1). $\tilde{\gamma}_\omega(t)$ follows a configuration $\zeta = (i_1, ...)$ with tangency on ∂D_{i_1}.

In the case (a1) if the ray $\tilde{\gamma}_\omega(t)$ meets an obstacle $D_j, j \neq i_s$, or go to infinity without reflections and tangency on ∂D, we obtain a contradiction with the choice of ω Indeed, for every sufficiently small $\eta > 0$ the ray $\tilde{\gamma}(t)$ issued from $v_1(\omega - \eta)$ must have reflection on ∂D_{i_s}. We apply the same argument for the case (b1).

We will show that the cases (a1), (b1) lead to contradiction.

Let $0 \leq t \leq d_2 T$ and assume that $\phi_t(v_1(\sigma)) \in \mathring{B} \cup \bar{\mathcal{D}}_{\text{in}}$ for $0 \leq \sigma_0 \leq \sigma \leq \sigma_1 \leq 1$. Therefore

$$\|\phi_t(v_1(\sigma_0)) - \phi_t(v_1(\sigma_1))\| = \| \int_{\sigma_0}^{\sigma_1} \frac{d}{ds}(\phi_t(v_1(\sigma))d\sigma\| \leq C_0 e^{\beta t}\|v_1(\sigma_0) - v_1(\sigma_1)\|$$

$$\leq C_0 e^{\beta d_2 T}\|\rho_1 - \rho\| \leq \frac{d_0}{2} e^{-\beta T} \leq \frac{d_0}{4}, \tag{4.5}$$

where we have used (4.1), (4.2), (4.3). Notice that if $\pi(\phi_t(v_1(\sigma))) \in \bar{\mathcal{D}}_{\text{in}}$, we take the incoming direction of $\phi_t(v_1(\sigma))$.

(a1). Let $\pi(\tilde{\gamma}_\omega(t))$ have a tangency at $v_s \in \partial D_{i_s}$ and let t_ω be the length of $\pi(\tilde{\gamma}_\omega(t))$ connecting $\pi(v_1(\omega)))$ and v_s We have the estimate

$$t_\omega \leq sd_1 < p_1 d_1 \leq \Big(\frac{2d_1}{d_0}\Big)T_1 = d_2 T_1.$$

Then, by using (4.5) with $t = t_\omega,\ s_0 = 0, s_1 = \omega$ one deduces

$$\|\pi(\phi_{t_\omega}(\rho_1)) - v_s\| \leq \frac{d_0}{4}.$$

Since $\tilde{\gamma}_1$ has period $T_1 \geq d_0$, obviously $\pi(\phi_{t_\omega}(\rho_1)) = \pi(\phi_{t_\omega+T_1}(\rho_1))$, and

$$\|\pi(\phi_{t_\omega+T_1}(\rho_1)) - \pi(\phi_{t_\omega+T_1}(v_1(\omega)))\| \geq \|\pi(\phi_{t_\omega+T_1}(v_1(\omega))) - v_s\|$$
$$-\|\pi(\phi_{t_\omega}(\rho_1)) - v_s\| \geq T_1 - \frac{d_0}{4} \geq \frac{3d_0}{4}.$$

Taking into account the estimate

$$\|\pi(\phi_{t_\omega+T_1}(\rho_1)) - \pi(\phi_{t_\omega+T_1}(v_1(\omega)))\|$$
$$\leq C_0 e^{\beta(t_\omega+T_1)}\|\rho_1 - \rho\| \leq C_0 e^{\beta(1+d_2)T_1}\|\rho_1 - \rho\| \leq C_0\epsilon_0 \leq \frac{d_0}{2},$$

we obtain a contradiction.

(b1). Let $v_1(\omega) = (\mu, \xi) \in \mathring{B}$. Consider the ray $\tilde{\zeta}_\omega(t)$ issued from $\nu = (\mu, -\xi) \in \mathring{B}$ with orientation inverse to that of $\tilde{\gamma}_\omega(t)$. There are two possibilities: (I). $\tilde{\zeta}_\omega(t)$ follows a configuration $\zeta = (z, ...),\ z \neq i_{i_1}$ with reflection or tangency at $v_z \in \partial D_z$, (II). $\tilde{\zeta}_\omega(t)$ does not meet ∂D and go to infinity.

In the case (I) let $\pi(\tilde{\gamma}_\omega)$ have tangency at $w_1 \in \partial D_{i_1}$ and let t_1 be the length of the segment connecting $v_1(\omega)$ and w_1. The ray $\tilde{\gamma}_\omega(t)$ after the tangency at w_1 goes to infinity. We repeat the argument of (a1) estimating $\|\pi(\phi_{t_1+T_1}(\rho_1)) - \pi(\phi_{t_1+T_1}(v_1(\omega)))\|$ and obtain a contradiction. In the case (II) we apply (4.5) with negative time $t = -T_1$ and $s_0 = 0, s_1 = \omega$ because we changed the orientation. Since $\phi_{-T_1}(\rho_1) = \rho_1$, we obtain

$$\|\pi(\phi_{-T_1}(\nu)) - \pi(\rho_1)\| \geq T_1 - \|\pi(\nu) - \pi(\rho_1)\| \geq d_0 - \epsilon_0 \geq \frac{3d_0}{4}$$

and

$$\|\pi(\phi_{-T_1}(\nu)) - \pi(\phi_{-T_1}(\rho_1))\| \leq C_0 e^{\beta T_1}\|\rho_1 - \rho\| \leq C_0\epsilon_0 e^{-\beta((1+d_2)T - T_1)}$$
$$\leq \frac{d_0}{4}$$

which leads to a contradiction.

Combining the analysis of the above cases, we deduce that the existence of $0 < \omega \leq 1$ is impossible. Thus we conclude that the ray $\tilde{\gamma}_\rho(t)$

issued from ρ follows the configuration β_1. We repeat the above argument for the periodic ray $\tilde{\gamma}_2(t)$ issued from ρ_2 and deduce that $\tilde{\gamma}_\rho(t)$ follows the configuration β_2. Since $\beta_1 \neq \beta_2$, this implies a contradiction with the assumption (4.4).

□

5. Open problems

The statement of Theorem 4.1 is true for obstacles satisfying (1.1). For perturbation arguments it is important to know if for every $\tilde{\gamma} \in \mathcal{G}(T)$ with $T \geq T_0$ and $x \in \pi(\tilde{\gamma}) \cap \partial D$ there exist $\alpha \gg 1$, $T_0 \gg 1$ and a neighbourhood

$$B(x, e^{-\alpha T}) = \{y \in \partial D : \ \|x - y\| \leq e^{-\alpha T}\}$$

such that

$$\forall \zeta \in \mathcal{G}(T) \setminus \tilde{\gamma}, \ B(x, e^{-\alpha T}) \cap \zeta = \emptyset. \tag{5.1}$$

In general this is not true since there are different periodic rays passing through a point $x \in \partial D$ with different directions (see [PS17, Section 2.1] for examples). On the other hand, in [PS17, Theorem 6.2.3] it was established that for generic obstacles for every $x \in \partial D$ there exists at most *one direction* $\xi \in \mathbb{S}^{d-1}$ (up to symmetry with respect to the normal to ∂D at x) such that (x, ξ) could generate a periodic ray. The reader may consult [PS17, Section 6.2]) for the precise definition of generic obstacles. Since there are only finite number periodic rays with period T, for generic obstacles every point $x \in \partial D$ has a suitably small neighbourhood with the property mentioned above. However, the size of these neighbourhoods could be extremely small and their dependence of T is unknown. We conjecture that there exist $\alpha \gg 1, T_0 \gg 1$ such that for generic obstacles for all $\zeta \in \mathcal{G}(T) \setminus \tilde{\gamma}, T \geq T_0$ the property (5.1) holds. For metrics on compact Riemannian manifolds with negative curvature a relation similar to (5.1) has been proved in [Sch20, Proposition 4]) without generic assumption.

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